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Dirac Equation in Metric-Affine Gravitation Theories and Superpotentials

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We consider a metric-affine gravitational framework in which the dynamical fields are the spin structures, the general linear connections, and the Dirac fermion fields. Using a spin structure and a linear connection on the world manifold, we construct a principal connection on the spinor bundle. By applying general ideas concerning the conservation laws in the Lagrangian approach to field theory, we examine the corresponding conserved currents. The main result is that the currents associated with infinitesimal vertical (internal) transformations of the covariance group are shown to vanish identically. It follows that to every vector field on the world manifold there corresponds a well-defined current, the stress-energy-momentum of the fields. It turns out that the fermion fields do not contribute at all to the superpotential terms. Actually the expression we get for the superpotential generalizes the well-known expression obtained by Komar.

1. INTRODUCTION

It is well known (Goldberg, 1980; Fletcher, 1960; Giachetta and Sardanashvily, 1995a,b) that all conservation laws

$$\partial_\lambda V^\lambda = 0 \tag{1}$$

which take place in generally covariant theories are *strong* laws, that is, when the field equations are satisfied, the current V^λ can be written as

$$V^\lambda = \partial_\mu U^{\lambda\mu} \tag{2}$$

where the skew-symmetric tensor density $U^{\lambda\mu}$ is called the *superpotential*. As an example, in the purely metric Einstein gravitation theory (Novotný,

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1984), the Einstein–Hilbert Lagrangian density $L_{EH} = \sqrt{-g}R$ leads to the well-known Komar superpotential (Komar, 1959)

$$U^{\lambda\mu}(\xi) = \sqrt{-g}(\xi^{\mu}_{;\alpha}g^{\lambda\alpha} - \xi^{\lambda}_{;\alpha}g^{\mu\alpha}) \tag{3}$$

where ξ is a vector field on the world manifold X and the subscript $;\alpha$ denotes covariant differentiation with respect to the Levi-Civita connection.

Recently, it was shown (Borowiec *et al.*, 1994) that the superpotential (3) has a kind of universal property, in the sense that the stress-energy-momentum tensor of any Lagrangian density depending on a metric and a symmetric connection through the scalar curvature reduces to the Komar superpotential. Giachetta and Sardanashvily (1995c) extended this result to the framework of the metric-affine gravitation theory in which the general linear connections replace the symmetric connections. In particular, it has been shown that a generally covariant Lagrangian density L leads to a current which is brought into the form (2) with

$$U^{\lambda\mu}(\xi) = \pi_{\alpha}{}^{\beta\mu\lambda}(\nabla_{\beta}\xi^{\alpha} + T^{\alpha}{}_{\sigma\beta}\xi^{\sigma}) \tag{4}$$

Here ∇ is the covariant derivative with respect to the linear connection K, T is the torsion of K, and

$$\pi_{\alpha}{}^{\beta\lambda\mu} = \partial L/\partial k^{\alpha}{}_{\beta\lambda,\mu}$$

are the momenta corresponding to the connection variables. This is a generalized Komar superpotential. Note that in the case of the Einstein–Hilbert Lagrangian density L_{EH}, we have

$$\frac{\partial L_{EH}}{\partial k^{\alpha}{}_{\beta\lambda,\mu}} = \sqrt{-g}(g^{\beta\mu}\delta^{\lambda}_{\alpha} - g^{\beta\lambda}\delta^{\mu}_{\alpha})$$

and the torsion vanishes. Hence (4) recovers the Komar superpotential (3).

The present paper is concerned with the conservation laws and the energy-momentum superpotentials in gravitation theories. We consider a metric-affine framework where the dynamical fields are the spin structures, the general linear connections on the world manifold X, and the Dirac fermion fields. We show that the choice of a spin structure and a linear connection on X allows the construction of a covariant derivative of fermion fields. Using this covariant derivative, we are able to write a Lagrangian density of fermion fields. We also consider a Lagrangian density of the gravitational fields. The total Lagrangian is then taken to be the sum of the two Lagrangians and is assumed to be generally covariant with respect to the group AUT($S(X)$) of principal automorphisms of the spin bundle $S(X)$.

The analysis of these questions is based on the *first variational formula* in Lagrangian field theory (Giachetta and Sardanashvily, 1995a; Mangiarotti and Modugno, 1983). In accordance with this formula, the invariance of a Lagrangian density under a general covariance group leads to conservation laws of currents which can be brought into the form of superpotentials. We show that the fermion fields do not contribute to the energy-momentum superpotential, which has exactly the form given in (4). Hence, one can think of the generalized Komar superpotential as being the universal energy-momentum superpotential of gravitational theories.

The organization of the paper is as follows. In Section 2 we recall the main facts about the Lagrangian formalism of field theory in jet bundle terms. Here the most important formula is the first variational formula of the calculus of variations, which plays a fundamental role in the study of symmetries and conservation laws of physical systems.

In Section 3 we introduce the configuration bundle of the metric-affine gravity interacting with fermion fields. Although we are talking about the metric-affine theory, we do not regard the metric of the world manifold as a proper dynamical field. Actually, in our construction the spin structures are taken to be true dynamical fields. However, every spin structure determines a metric on the world manifold X. The most important point in this section is the construction of the covariant derivative of spinor fields using a spin structure and a linear connection on X.

In Section 4 we describe the actions of AUT($S(X)$) on the various bundles involved in the theory. We also find the expression of the vector fields corresponding to the infinitesimal version of these actions.

Finally, in Section 5 we consider a Lagrangian density which has the group AUT($S(X)$) of principal automorphisms of the spin bundle $S(X)$ as the general covariance group. We find the corresponding currents and show that they can be brought into the form of a generalized Komar superpotential.

2. CALCULUS OF VARIATIONS

In this section we briefly introduce the basic features of the geometric approach to Lagrangian field theory. Accordingly, classical fields are represented by sections of a bundle $Y \to X$ over a world manifold X and their dynamics is phrased in terms of jet manifolds (Sardanashvily, 1993). We restrict ourselves to the first-order Lagrangian formalism, since this is enough for our purposes. Here Y is the configuration space of fields described by sections s: $X \to Y$, whereas the first-order jet manifold J^1Y of $Y \to X$ is the phrase space.

Roughly speaking, one can say that the k-order jet manifold J^kY of a bundle $Y \to X$ comprises the equivalence classes $j^k_x s$, $x \in X$, of sections s: $X \to Y$ identified by the first $k + 1$ terms of their Taylor series at a point x. Let (x^λ, y^i) be fibered coordinates on Y, with $1 \leq \lambda \leq m = \dim X$ and $1 \leq$

$i \leq n$, $m + n = \dim Y$. The induced coordinates on J^1Y are denoted by $(x^\lambda, y^i, y^i_\lambda)$. Their meaning is clear; given a section s: $X \to Y$, let j^1s: $X \to J^1Y$ denote its first-order jet extension. Then we have

$$y^i \circ s = s^i, \qquad y^i_\lambda \circ j^1 s = \partial_\lambda s^i$$

As usual, the coordinate fields associated with $(x^\lambda, y^i, y^i_\lambda)$ are denoted by $(\partial_\lambda, \partial_i, \partial^\lambda_i)$.

A basic operation on jet spaces is the following. Let

$$u\colon \quad Y \to TY, \qquad u = u^\lambda(x)\partial_\lambda + u^i(x, y)\partial_i$$

be a projectable vector field on Y representing an infinitesimal transformation of both the field and the world manifold variables y^i and x^λ, respectively. Then u can be lifted to a (projectable) vector field $\bar{u}$ on J^1Y given by

$$\begin{aligned} &\bar{u}\colon \quad J^1Y \to TJ^1Y \\ &\bar{u} = u^\lambda \partial_\lambda + u^i \partial_i + u^i_\lambda \partial^\lambda_i \\ &u^i_\lambda = J_\lambda u^i - y^i_\mu \partial_\lambda u^\mu \end{aligned} \tag{5}$$

where

$$J_\lambda = \partial_\lambda + y^j_\lambda \partial_j + \cdots$$

is the total derivative with respect to x^λ.

Let

$$\begin{aligned} &\mathcal{L}\colon \quad J^1Y \to \overset{m}{\wedge} T^*X \\ &\mathcal{L} = L(x^\lambda, y^i, y^i_\lambda)\omega, \qquad \omega = dx^1 \wedge \cdots \wedge dx^m \end{aligned} \tag{6}$$

be a first-order *Lagrangian density*. Then we have the following objects naturally associated with it. The *Poincaré–Cartan form*

$$\begin{aligned} &\Xi(\mathcal{L})\colon \quad J^1Y \to \overset{m}{\wedge} T^*Y \\ &\Xi(\mathcal{L}) = \mathcal{L} + \partial^\lambda_i L(dy^i - y^i_\mu dx^\mu) \wedge \omega^\lambda, \qquad \omega^\lambda = \partial_\lambda \rfloor \omega \end{aligned} \tag{7}$$

which is the unique Lepagian (Cartan) form equivalent to $\mathcal{L}$ in the first-order case. The *Euler–Lagrange operator*

$$\begin{aligned} &\mathcal{E}(\mathcal{L})\colon \quad J^2Y \to V^*Y \otimes \overset{m}{\wedge} T^*X \\ &\mathcal{E}(\mathcal{L}) = \delta_i L \, dy^i \otimes \omega, \qquad \delta_i L = \partial_i L - J_\lambda(\partial^\lambda_i L) \end{aligned} \tag{8}$$

whose kernel consists of critical sections s: $X \to Y$ of the variational problem defined by the Lagrangian density $\mathcal{L}$.

If $u\colon Y \to TY$ is a projectable vector field, we define the corresponding *current* to be

$$V(\mathcal{L}, u)\colon \quad J^1Y \to \overset{m-1}{\wedge} T^*X, \qquad V(\mathcal{L}, u) = V^\lambda(\mathcal{L}, u)\omega_\lambda \tag{9}$$

$$V^\lambda(\mathcal{L}, u) = \partial_i^\lambda L(u^i - y^i_\mu u^\mu) + u^\lambda L$$

This is defined as the horizontal projection of the $(m - 1)$-form $u \rfloor \Xi(\mathcal{L})$.

There are different methods to discover differential conservation laws in Lagrangian field theories (Fletcher, 1960). Here we are concerned with the so-called *symmetry method.* Let $u\colon Y \to TY$ be a projectable vector field on Y. By computing the Lie derivative $L_{\bar{u}}\mathcal{L}$ of the Lagrangian density $\mathcal{L}$ by the lift of u, we find the relation

$$L_{\bar{u}}\mathcal{L} = u_V \rfloor \mathcal{E}(\mathcal{L}) + d_H V(\mathcal{L}, u) \tag{10}$$

This is a basic formula known as the *first variational formula* of the calculus of variations. On the right-hand side of (10),

$$u_V\colon \quad J^1Y \to VY, \qquad u_V = (u^i - y^i_\lambda u^\lambda)\partial_i$$

is the vertical part of the vector field u and

$$d_H V(\mathcal{L}, u)\colon \quad J^2Y \to \overset{m}{\wedge} T^*X,$$

$$J_\lambda V^\lambda(\mathcal{L}, u)\omega$$

is the horizontal derivative of $V(\mathcal{L}, u)$.

Now assume that the vector field u is an infinitesimal *symmetry transformation* of the Lagrangian density $\mathcal{L}$, i.e.,

$$L_{\bar{u}}\mathcal{L} = 0 \tag{11}$$

Then the variational formula (10) yields the conservation law

$$\partial_\lambda(V^\lambda(\mathcal{L}, u) \circ j^1 s) = 0 \tag{12}$$

when the field equations $\mathcal{E}(\mathcal{L}) \circ j^2 s = 0$ are satisfied. These are called *weak* conservation laws. However, if the field u is an infinitesimal transformation belonging to a function group which is a symmetry group of the Lagrangian density $\mathcal{L}$, such as gauge theories and general relativity, then the current $V(\mathcal{L}, u)$ takes the form

$$V(\mathcal{L}, u) = W(\mathcal{L}, u) + d_H U(\mathcal{L}, u) \tag{13}$$

where $W(\mathcal{L}, u)$ vanishes on solutions of the field equations and

$$U(\mathcal{L}, u)\colon \quad J^1Y \to \overset{m-2}{\wedge} T^*X$$

is a $(m - 2)$-form called the *superpotential*. The corresponding conservation law (12) is called a *strong* conservation law.

3. METRIC-AFFINE GRAVITY: KINEMATICS

Hereafter, the 4-dimensional base manifold X is assumed to satisfy the well-known topological conditions which guarantee the existence of a Lorentzian metric and a spin structure. We summarize these conditions by assuming that the manifold X is not compact and its tangent bundle is trivial. We call X the world manifold. The Lorentzian metrics and the general linear connections on X are called the world metrics and the world connections, respectively.

We use three kinds of indices. Greek indices label points of the world manifold. Lowercase Latin indices are reserved for quantities defined on the Minkowski space-time. Both kinds of indices range from 0 to 3. Uppercase Latin indices are used for the spinor fields and range from 1 to 4.

In the gauge gravitation theory, gravity is described by pairs (h, A_h) of gravitational fields h and associated Lorentz connections A_h (Sardanashvily and Zakharov, 1992). The Lorentz connection A_h is usually used to construct the covariant derivative of Dirac fermion fields in the presence of the gravitational field h. On the other hand, in the metric-affine gravitation theory (Hehl *et al.*, 1995; Aringazin and Mikhailov, 1991; Tucker and Wang, 1995) the connection is no longer assumed in advance to be a Lorentz connection, though this may result from the field equations. Hence, the problem of constructing a covariant derivative of spinor fields arises.

Let $L(X)$ be the principal fiber bundle (PFB) of oriented linear coframes on X. In gravitation theory, its structure group $GL^+(4, R)$ reduces to the connected Lorentz group $L \subset SO(1, 3)$. This means that there exists a reduced subbundle $L^h(X)$ of $L(X)$ whose structure group is L. As is well known (Kobayashi and Nomizu, 1963), there is a 1:1 correspondence between the reduced L-subbundles $L^h(X)$ of $L(X)$ and the Lorentzian metrics on X of signature $(1, -1, -1, -1)$.

Now we give the following definition (Van der Neuvel, 1994).

Definition 3.1. Let $\rho: L_s \to L$ be the universal covering morphism, where $L_s = SL(2, C)$. A spin structure on X consists of a PFB $\pi_s: S(X) \to X$ with structure group L_s and a map $h: S(X) \to L(X)$ satisfying the following properties:

(i) $h \circ R_A = R_{\rho(A)} \circ h$, $\forall A \in L_s$.

(ii) $\pi \circ h = \pi_s$.

Notice that a spin structure determines a reduction of $L(X)$ to an L-subbundle $L^h(X)$ and, hence, a metric g on X. This subbundle is the image of $S(X)$ under h: $S(X) \to L(X)$. Hereafter, we choose a PFB π_s: $S(X) \to X$ and consider the spin structures provided by all the morphisms h: $S(X) \to L(X)$ which satisfy the properties (i) and (ii) of Definition 3.1.

These morphisms can also be seen as sections of a fiber bundle $S \to X$ defined by

$$S = \bigcup_{x \in X} S_x \tag{14}$$

$$S_x = \{h_x: S(X)_x \to L(X)_x \mid h_x(p \cdot A) = h_x(p) \cdot \rho(A), \forall p \in S(X)_x, A \in L_x\}$$

Fibered coordinates on S are introduced as follows. Let σ: $U \to S(X)$ and (dx^λ): $U \to L(X)$ be local gauges over the same neighborhood U of X. Then, for any $h_x \in S$ with $x \in U$, we have

$$h_x \circ \sigma(x) = (h^a), \qquad h^a = h^a_\lambda \, dx^\lambda \tag{15}$$

where the matrix $(h^a{}_\lambda) \in GL^+(4, R)$. Hence the coordinates of h_x are taken to be $(x^\lambda, h^a{}_\lambda)$.

The world connections are principal connections on the PFB $L(X)$. It follows that there is a 1:1 correspondence between world connections and global sections of the quotient bundle (Giachetta and Mangiarotti, 1990)

$$C = J^1L(X)/GL^+(4, R) \to X \tag{16}$$

With respect to a holonomic gauge (dx^λ): $U \to L(X)$, the bundle C is coordinatized by $(x^\lambda, k^\alpha{}_{\beta\lambda})$ so that, for any section K: $X \to C$,

$$K^\alpha{}_{\beta\lambda} = k^\alpha{}_{\beta\lambda} \circ K$$

and the connection parameters on T^*X, i.e.,

$$\nabla_\lambda \, dx^\alpha = K^\alpha{}_{\beta\lambda} \, dx^\beta \tag{17}$$

There are different ways to introduce Dirac fermion fields. Here we follow the algebric approach. Let (M, η) be a Minkowski space, with $\eta = \mathrm{diag}(1, -1, -1, -1)$, and $Cl_{1,3}$ the complex Clifford algebra generated by elements of M. The spinor space V is defined to be a minimal left ideal of $Cl_{1,3}$ on which this algebra acts on the left. We have a representation

$$\gamma: \quad M \otimes V \to V \tag{18}$$

of elements of the Minkowski space M by Dirac γ-matrices on V and the spinor representation

$$\mu: \quad L_s \to GL(V) \tag{19}$$

Then, the spinor bundle

$$F = (S(X) \times V)/L_s$$

is associated with the PFB π_s: $S(X) \to X$. Sections ψ: $X \to F$ represent Dirac fermion fields. With respect to a gauge σ: $U \to S(X)$ and a basis (e_A) of V, the induced fibered coordinates on F are denoted by (x^λ, y^A).

We take the configuration space of the metric-affine gravitation theory in interaction with fermion fields to be the fibered product bundle

$$Q = S \times_X C \times_X F \to X \tag{20}$$

Let us consider a bundle of complex Clifford algebras $Cl_{1,3}$ over X whose structure group is the Clifford group of invertible elements of $Cl_{1,3}$. It has the subbundle $Y_M \to X$ of Minkowski spaces of generating elements of $Cl_{1,3}$. To describe Dirac fermion fields on a world manifold X, one must require Y_M to be isomorphic to the cotangent bundle T^*X of X. It takes place if there exists a reduced L-subbundle $L^h(X)$ of $L(X)$ such that

$$Y_M = (L^h(X) \times M)/L$$

The representation (18) satisfies the following equivariance property:

$$\gamma(\rho(A)\cdot\xi \otimes \mu(A)\cdot v) = \mu(A)\cdot(\gamma(\xi \otimes v)), \qquad \forall A \in L_s, \quad \xi \in M, \quad v \in V \tag{21}$$

Owing to this property, the map γ goes to the quotient and defines the representation

$$\gamma_h: \quad T^*X \otimes F = (S(X) \times M \otimes V)/L_s \to (S(X) \times V)/L_s = F \tag{22}$$

of cotangent vectors to the world manifold X by Dirac γ-matrices on elements of the spinor bundle F. If σ: $U \to S(X)$ is a local gauge, (e_A) is a basis of V, $(\hat{e}_A)$ is the induced basis of the free module of fermion fields over U, and $(h^a) = h \circ \sigma$: $U \to L(X)$ is the image of σ under the spin structure h, then we have

$$\gamma_h(dx^\lambda \otimes \hat{e}_A) = [\sigma, h^\lambda{}_a \gamma^{aB}{}_A] \tag{23}$$

where $dx^\lambda = h^\lambda{}_a h^a$ and the square bracket denotes equivalence classes in the quotient space.

We shall say that sections of the spinor bundle $F \to X$ describe Dirac fermion fields in the presence of the world metric induced by the spin structure h. Indeed, given a principal connection A on F, let ∇ be the corresponding covariant differential operator. Then, using the representation (22), one can construct the Dirac operator

$$D_h = \gamma_h \circ \nabla: \quad J^1F \to T^*X \otimes F \to F \tag{24}$$

$$y^A \circ D_h = h^\lambda{}_a \gamma^{aA}{}_B(y^B_\lambda - \tfrac{1}{2}A^{ab}_\lambda I_{ab}{}^B{}_C y^C)$$

on the spinor bundle F. Here $A^{ab}_{\lambda} = -A^{ba}_{\lambda}$ are the connection parameters of A (which are opposite in sign to the gauge potentials) and

$$I_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$$

are the generators of the spinor group L_s. Different spin structures h and h' yield nonequivalent representations γ_h and $\gamma_{h'}$. It follows that a Dirac fermion field must be regarded only in a pair with a certain spin structure.

In order to write the Lagrangian density of fermion fields in the metric-affine framework, we have to construct a principal connection on F starting from a world connection and a spin structure. Let $\omega: L(X) \to T^*L(X) \otimes gl(4, R)$ be a connection one-form and $h: S(X) \to L(X)$ a spin structure. Note that $gl(4, R) = \ell + m$ (direct sum), where ℓ is the Lie algebra of L and m is a subspace of $gl(4, R)$ such that $\Lambda \cdot m \cdot \Lambda^{-1} \subset m$, for any $\Lambda \in$ L. Then, by a well-known theorem (Kobayashi and Nomizu, 1963, Proposition 6.4, p. 83), the pullback by h of the ℓ-component ω' of ω defines a principal connection $\omega_h = h^*\omega'$ on $S(X)$. This is the principal connection on F we were looking for.

The connection parameters of ω_h can be found as follows. The coordinate expression of ω in a bundle chart $(x^\lambda, h^a{}_\lambda)$ of $L(X)$ with respect to a holonomic coframe (dx^λ) is

$$\omega^a{}_b = h^a{}_\alpha(dh^\alpha{}_b - K^\alpha{}_{\beta\lambda}h^\beta{}_b\, dx^\lambda) \tag{25}$$

where the connection parameters $K^\alpha{}_{\beta\lambda}$ are defined in (17). Projecting ω on the Lie algebra of L leads to

$$\omega^{[ab]} = \frac{1}{2}(h^a{}_\alpha\eta^{bc} - h^b{}_\alpha\eta^{ac})(dh^\alpha{}_c - K^\alpha{}_{\beta\lambda}h^\beta{}_c\, dx^\lambda) \tag{26}$$

Now let $\sigma: U \to S(X)$ be a local gauge and let $h \circ \sigma = (h^a)$, $h^a = h^a{}_\lambda\, dx^\lambda$. It follows that the connection parameters of ω_h are given by

$$A^{ab}_\lambda = -\frac{1}{2}(h^a{}_\alpha\eta^{bc} - h^b{}_\alpha\eta^{ac})(\partial_\lambda h^\alpha{}_c - K^\alpha{}_{\beta\lambda}h^\beta{}_c) \tag{27}$$

4. BASIC REPRESENTATIONS

Let Diff($S(X)$) be the group C^∞ diffeomorphisms of $S(X)$. We shall consider the following infinite-dimensional groups:

$$\mathrm{AUT}(S(X)) = \{\Phi \in \mathrm{Diff}(S(X)) \mid \Phi(p\cdot A) = \Phi(p)\cdot A\ \forall p \in S(X), A \in L_s\}$$

$$\mathrm{Aut}(S(X)) = \{\Phi \in \mathrm{AUT}(S(X)) \mid \Phi \text{ covers the identity of } X\}$$

$$\mathrm{Diff}(X) = \{\phi: X \to X \mid \phi \text{ is a } C^\infty\text{-diffeomorphism}\}$$

We denote by $\hat{\pi}_s$: AUT($S(X)$) $\to$ Diff(X) the group morphism that takes an automorphism of $S(X)$ to its induced diffeomorphism of X. Of course, its kernel is the subgroup Aut($S(X)$).

A principal automorphism $\Phi \in \mathrm{AUT}(S(X))$, with $\hat{\pi}_s(\Phi) = \phi$, induces automorphisms $\hat{\phi}$, Φ_S, Φ_C, and Φ_F of the bundles $L(X)$, S, C, and F, all covering ϕ. We begin by defining $\hat{\phi}$ as

$$\hat{\phi}((h^a)) = ((T\phi^{-1})^*(h^a)), \qquad \forall (h^a) \in L(X) \tag{28}$$

One can easily verify that $\hat{\phi}$ is equivariant with respect to the right action of $GL^+(4, R)$ on $L(X)$. Due to this property, the jet extension $j^1\hat{\phi}$: $J^1L(X) \to J^1L(X)$ of $\hat{\phi}$ goes to the quotient $C = J^1L(X)/GL^+(4, R)$ and defines an automorphism

$$\phi_C\colon \quad C \to C \tag{29}$$

By recalling that $F = (S(X) \times V)/L_s$ is a bundle associated with $S(X)$, we define

$$\begin{aligned} &\Phi_F\colon \quad F \to F \\ &\Phi_F([p, v]) = [\Phi(p), v], \qquad \forall p \in S(X), \quad v \in V \end{aligned} \tag{30}$$

Finally, we define

$$\begin{aligned} &\Phi_S\colon \quad S \to S \\ &\Phi_S(h_x) = \hat{\phi} \circ h_x \circ \Phi^{-1} \in S_{\phi(x)}, \qquad \forall h_x \in S_x \end{aligned} \tag{31}$$

We shall also consider the infinitesimal version of these actions of $\mathrm{AUT}(S(X))$ on S, C, and F. Let $T_{L_s}S(X) = TS(X)/L_s$ be the quotient of the tangent bundle $TS(X)$ by the spinor group L_s. This is a vector bundle over X whose sections ξ: $X \to T_{L_s}S(X)$ are L_s-invariant vector fields on $S(X)$. In particular, they are projectable fields, that is, every section ξ: $X \to T_{L_s}S(X)$ induces a vector field ξ_X on X. A subbundle of $T_{L_s}S(X)$ is $V_{L_s}S(X) = VS(X)/L_s$, where $VS(X) \subset TS(X)$ is the bundle of vertical vectors. Sections ξ: $X \to V_{L_s}S(X)$ of this bundle are L_s-invariant vertical vector fields on $S(X)$ (Giachetta and Mangiarotti, 1990).

Let ξ: $X \to T_{L_s}S(X)$ be a section. Its flow Φ_t, with $\hat{\pi}_s(\Phi_t) = \phi_t$, defines a one-parameter group of principal automorphisms of $S(X)$ covering $\phi_t \in \mathrm{Diff}(X)$. Then the corresponding one-parameter groups of automorphisms induced on S, C, and F define vector fields ξ_S, ξ_C, and ξ_F on S, C, and F, respectively, and hence on the configuration space Q. If

$$\xi = \xi^\lambda \partial_\lambda + \tfrac{1}{2}\xi^{ab} e_{ab}, \qquad \xi_X = \xi^\lambda \partial_\lambda$$

is the coordinate expression of the section ξ: $X \to T_{L_s}S(X)$, where (e_{ab}) is a basis of the Lie algebra of L, then their local expressions are

$$\xi_S: \quad S \to TS$$

$$\xi_S = \xi^\lambda \partial_\lambda + (h^\alpha{}_a \partial_\alpha \xi^\lambda - h^\lambda{}_b \xi^b{}_a) \frac{\partial}{\partial h^\lambda{}_a}$$

$$\xi_C: \quad C \to TC$$

$$\xi_C = \xi^\lambda \partial_\lambda + (\partial_{\mu\nu} \xi^\lambda - k^\lambda{}_{\mu\alpha} \partial_\nu \xi^\alpha - k^\lambda{}_{\alpha\nu} \partial_\mu \xi^\alpha + k^\alpha{}_{\mu\nu} \partial_\alpha \xi^\lambda) \frac{\partial}{\partial k^\lambda{}_{\mu\nu}}$$

$$\xi_F: \quad F \to TF$$

$$\xi_F = \xi^\lambda \partial_\lambda + \frac{1}{2} \xi^{ab} I_{ab}{}^A{}_B y^B \frac{\partial}{\partial y^A} + \frac{1}{2} \xi^{ab} \bar{y}_A I^{+A}_{ab}{}_B \frac{\partial}{\partial \bar{y}_B}$$

Later we shall use the compact notation $y^i = k^\lambda{}_{\mu\nu}$ and

$$\xi_C = \xi^\lambda \partial_\lambda + (u^{i\mu\nu}_\lambda \partial_{\mu\nu} \xi^\lambda + u^{i\lambda}_\mu \partial_\lambda \xi^\mu) \frac{\partial}{\partial y^i}$$

Finally, we have

$$\xi_Q: \quad Q \to TQ$$

$$\xi_Q = \xi^\lambda \partial_\lambda + (h^\alpha{}_a \partial_\alpha \xi^\lambda - h^\lambda{}_b \xi^b{}_a) \frac{\partial}{\partial h^\lambda{}_a} + (u^{i\mu\nu}_\lambda \partial_{\mu\nu} \xi^\lambda + u^{i\lambda}_\mu \partial_\lambda \xi^\mu) \frac{\partial}{\partial y^i} \tag{32}$$

$$+ \frac{1}{2} \xi^{ab} I_{ab}{}^A{}_B y^B \frac{\partial}{\partial y^A} + \frac{1}{2} \xi^{ab} \bar{y}_A I^{+A}_{ab}{}_B \frac{\partial}{\partial \bar{y}_B}$$

5. SYMMETRIES AND CONSERVATION LAWS

Let us consider a first-order Lagrangian density $\mathcal{L}: J^1Q \to \overset{m}{\wedge} T^*X$ on the configuration space Q, (20), which is the sum of a Lagrangian density $\mathcal{L}_f$ of fermion fields and a Lagrangian density $\mathcal{L}_g$ of the metric-affine gravity.

To construct $\mathcal{L}_f$ we use the representation (22) and the principal connection (27). Moreover, we need a real-valued fiber metric k on the spinor bundle F. We take it to be that induced by the L_s-invariant metric $\hat{k}: V \times V \to R$ given by

$$\hat{k}(v, w) = \frac{1}{2} (w^+ \gamma^0 v + v^+ \gamma^0 w)$$

Then, we define $\mathcal{L}_f$ as

$$\mathcal{L}_f(h, K, \psi) = [k(iD_h \psi, \psi) - mk(\psi, \psi)] \mu_h \tag{33}$$

where $\mu_h: X \to \overset{m}{\wedge} T^*X$ is the metric volume form induced by the spin structure h. Its local expression is

$$
\begin{aligned}
\mathcal{L}_f &= L_f \omega \\
L_f &= \frac{i}{2} [\bar{y}_A (\gamma^0 \gamma^\lambda)^A{}_B (y^B_\lambda - \tfrac{1}{2} A^{ab}_\lambda I_{ab}{}^B{}_C y^C) \\
&\quad - (\bar{y}_{A\lambda} - \tfrac{1}{2} A^{ab}{}_\lambda \bar{y}_C I^{+}_{ab}{}^C{}_A)(\gamma^0 \gamma^\lambda)^A{}_B y^B] \det(h^a{}_\lambda) \\
&\quad - m \bar{y}_A (\gamma^0)^A{}_B y^B \det(h^a{}_\lambda)
\end{aligned} \tag{34}
$$

where the connection parameters A^{ab}_λ are given in (27) and $\gamma^\lambda = \gamma^a h^\lambda{}_a$.

One can easily verify that

$$
\frac{\partial L_f}{\partial k^\lambda{}_{\mu\nu}} + \frac{\partial L_f}{\partial k^\lambda{}_{\nu\mu}} = 0 \tag{35}
$$

Hence, the Lagrangian density (33) depends only on the torsion of the world connection K. In particular, it follows that if K is the Levi-Civita connection of the gravitational field associated with the spin structure h, then $\mathcal{L}_f$ takes the form of the familiar Lagrangian density of fermion fields in Einstein's gravitation theory.

Now let $h: X \to S$, $K: X \to C$, and $\psi: X \to F$ be sections. For any $\Phi \in \text{AUT}(S(X))$, covering $\phi \in \text{Diff}(X)$, we define $\tilde{h} = \phi_S \circ h \circ \phi^{-1}$, $\tilde{K} = \Phi_C \circ K \circ \phi^{-1}$, and $\tilde{\psi} = \Phi_F \circ \psi \circ \phi^{-1}$. Then, it is easily verified that the Lagrangian density (33) is *generally covariant* with respect to $\text{AUT}(S(X))$, that is

$$
\mathcal{L}_f(\tilde{h}, \tilde{K}, \tilde{\psi}) = (\phi^{-1})^* \mathcal{L}_f(h, K, \psi) \tag{36}
$$

for any $\Phi \in \text{AUT}(S(X))$. It follows that, for any section $\xi: X \to T_{L_s}S(X)$, the Lie derivative of $\mathcal{L}_f$ by the jet lift $\bar{\xi}_Q$ of the vector field (32) is equal to zero, i.e.,

$$
L_{\bar{\xi}_Q} \mathcal{L}_f = 0, \qquad \forall \xi: \quad X \to T_{L_s}S(X) \tag{37}
$$

The Lagrangian density $\mathcal{L}_g$ of the metric-affine gravity is assumed to be of the form

$$
\begin{aligned}
\mathcal{L}_g &= L_g \omega \\
L_g &= f(g^{\lambda\mu}(h), R^\alpha{}_{\beta\lambda\mu}(K))
\end{aligned} \tag{38}
$$

where f is a scalar density,

$$
g^{\lambda\mu} = h^\lambda{}_a h^\mu{}_b \eta^{ab} \tag{39}
$$

is the metric induced by the spin structure h and

$$R^{\alpha}{}_{\beta\lambda\mu} = k^{\alpha}{}_{\beta\lambda,\mu} - k^{\alpha}{}_{\beta\mu,\lambda} + k^{\alpha}{}_{\nu\lambda}k^{\nu}{}_{\beta\mu} - k^{\alpha}{}_{\nu\mu}k^{\nu}{}_{\beta\lambda}$$

is the curvature tensor of the world connection K. In other words, we suppose that $\mathcal{L}_g$ factorizes through the curvature tensor and the combination (39). Then, one can easily prove the relations

$$\pi_{\alpha}{}^{\beta\lambda\mu} = -\pi_{\alpha}{}^{\beta\mu\lambda} \tag{40}$$

$$\partial L_g/\partial k^{\alpha}{}_{\beta\lambda} = \pi_{\sigma}{}^{\beta\lambda\nu}k^{\sigma}{}_{\alpha\nu} - \pi_{\alpha}{}^{\sigma\lambda\nu}k^{\beta}{}_{\sigma\nu} \tag{41}$$

We also assume that $\mathcal{L}_g$ is generally covariant, so that

$$L_{\bar{\xi}_Q}\mathcal{L}_g = 0, \qquad \forall\xi\colon\ X \to T_{L_s}S(X) \tag{42}$$

Let us apply now the machinery of Section 2 to the Lagrangian density $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_f$. We get the current

$$\begin{aligned} V^{\lambda}(\mathcal{L}, \xi) = {} & \frac{\partial L_g}{\partial y^i_{\lambda}} (u^{i\beta\gamma}_{\alpha}\partial_{\beta\gamma}\xi^{\alpha} + u^{i\alpha}_{\beta}\partial_{\alpha}\xi^{\beta} - u^{i}{}_{,\alpha}\xi^{\alpha}) \\ & + \frac{\partial L_f}{\partial h^{\alpha}{}_{c,\lambda}} (h^{\beta}{}_{c}\partial_{\beta}\xi^{\alpha} - h^{\alpha}{}_{a}\xi^{a}{}_{c} - h^{\alpha}{}_{c,\beta}\xi^{\beta}) \\ & + \frac{\partial L_f}{\partial y^A_{\lambda}} \left(\frac{1}{2}\,\xi^{ab}I_{ab}{}^{A}{}_{B}y^{B} - y^{A}_{\alpha}\xi^{\alpha}\right) \\ & + \frac{\partial L_f}{\partial \bar{y}_{A\lambda}} \left(\frac{1}{2}\,\xi^{ab}\bar{y}_{B}I^{+}_{ab}{}^{B}{}_{A} - \bar{y}_{A\alpha}\xi^{\alpha}\right) \\ & + \xi^{\alpha}(L_g + L_f) \end{aligned} \tag{43}$$

In particular, if $\xi\colon X \to V_{L_s}S(X)$ is a vertical field, i.e.,

$$\xi = \frac{1}{2}\,\xi^{ab}e_{ab}$$

then the current (43) reduces to

$$\begin{aligned} V^{\lambda}(\mathcal{L}, \xi) = {} & -\frac{\partial L_f}{\partial h^{\alpha}{}_{c,\lambda}}\, h^{\alpha}{}_{a}\xi^{a}{}_{c} + \frac{1}{2}\frac{\partial L_f}{\partial y^A_{\lambda}}\,\xi^{ab}I_{ab}{}^{A}{}_{B}y^{B} \\ & + \frac{1}{2}\frac{\partial L_f}{\partial \bar{y}_{A\lambda}}\,\xi^{ab}\bar{y}_{B}I^{+}_{ab}{}^{B}{}_{A} \end{aligned} \tag{44}$$

By explicit calculation we get

$$\begin{aligned} V^{\lambda}(\mathcal{L}, \xi) = {} & -\frac{i}{4}\,[\bar{y}_{A}(\gamma^{0}\gamma^{\lambda})^{A}{}_{B}A^{abc}_{\mu}I_{ab}{}^{B}{}_{C}y^{C} \\ & - A^{abc}_{\mu}\bar{y}_{C}I^{+}_{ab}{}^{C}{}_{A}(\gamma^{0}\gamma^{\lambda})^{A}{}_{B}y^{B}]h^{\mu}{}_{d}\xi^{d}{}_{c} \\ & + \frac{i}{4}\,[\bar{y}_{B}(\gamma^{0}\gamma^{\lambda})^{B}{}_{A}\xi^{ab}I_{ab}{}^{A}{}_{C}y^{C} - (\gamma^{0}\gamma^{\lambda})^{B}{}_{C}y^{C}\xi^{ab}\bar{y}_{A}I^{+}_{ab}{}^{A}{}_{B}] \end{aligned}$$

where we have set $A^{abc}_{\mu} = \frac{1}{2}(h^a{}_{\mu}\eta^{bc} - h^b{}_{\mu}\eta^{ac})$. Since

$$A^{abc}_{\mu}h^{\mu}{}_d\xi^d{}_c = \xi^{ab}$$

the above expression vanishes identically. It follows that the currents associated with vertical (internal) symmetries do not contribute to the general expression (43), which therefore takes the form

$$\begin{aligned} V^{\lambda}(\mathcal{L}, \xi) = {} & \frac{\partial L_g}{\partial y^i_{\lambda}} (u^{i\beta\gamma}_{\alpha}\partial_{\beta\gamma}\xi^{\alpha} + u^{i\alpha}_{\beta}\partial_{\alpha}\xi^{\beta} - y^i_{\alpha}\xi^{\alpha}) \\ & + \frac{\partial L_f}{\partial h^{\alpha}{}_{c,\lambda}} (h^{\beta}{}_c\partial_{\beta}\xi^{\alpha} - h^{\alpha}{}_{c,\beta}\xi^{\beta}) \\ & - \frac{\partial L_f}{\partial y^A_{\lambda}} y^A_{\alpha}\xi^{\alpha} - \frac{\partial L_f}{\partial \bar{y}_{A\lambda}} \bar{y}_{A\alpha}\xi^{\alpha} + \xi^{\alpha}(L_g + L_f) \end{aligned} \tag{45}$$

Due to the arbitrariness of the functions ξ^{α}, the equalities (37) and (42) imply the conditions

$$\delta^{\alpha}_{\lambda}L_g + h^{\alpha}{}_a \frac{\partial L_g}{\partial h^{\lambda}{}_a} + u^{i\alpha}_{\lambda} \frac{\partial L_g}{\partial y^i} + J_{\mu}u^{i\alpha}_{\lambda} \frac{\partial L_g}{\partial y^i_{\mu}} - y^i_{\lambda} \frac{\partial L_g}{\partial y^i_{\alpha}} = 0 \tag{46}$$

and

$$\begin{aligned} & \delta^{\alpha}_{\lambda}L_f + h^{\alpha}{}_a \frac{\partial L_f}{\partial h^{\lambda}{}_a} + h^{\alpha}{}_{a,\mu} \frac{\partial L_f}{\partial h^{\lambda}{}_{a,\mu}} - h^{\mu}{}_{a,\lambda} \frac{\partial L_f}{\partial h^{\mu}{}_{a,\alpha}} \\ & \quad + u^{i\alpha}_{\lambda} \frac{\partial L_f}{\partial y^i} - y^A_{\lambda} \frac{\partial L_f}{\partial y^A_{\alpha}} - \bar{y}_{A\lambda} \frac{\partial L_f}{\partial \bar{y}_{A\alpha}} = 0 \end{aligned} \tag{47}$$

Substituting the terms

$$\delta^{\alpha}_{\lambda}L_g - y^i_{\lambda} \frac{\partial L_g}{\partial y^i_{\alpha}}$$

and

$$\delta^{\alpha}_{\lambda}L_f - h^{\mu}{}_{a,\lambda} \frac{\partial L_f}{\partial h^{\mu}{}_{a,\alpha}} - y^A_{\lambda} \frac{\partial L_f}{\partial y^A_{\alpha}} - \bar{y}_{A\lambda} \frac{\partial L_f}{\partial \bar{y}_{A\alpha}}$$

from (46) and (47) into the conservation law (45), we find

$$\begin{aligned} V^{\lambda}(\mathcal{L}, \xi) = {} & \frac{\partial L_g}{\partial y^i_{\lambda}} (u^{i\beta\gamma}_{\alpha}\partial_{\beta\gamma}\xi^{\alpha} + u^{i\alpha}_{\beta}\partial_{\alpha}\xi^{\beta}) \\ & - \xi^{\alpha}\left(h^{\lambda}{}_a \frac{\partial L_g}{\partial h^{\alpha}{}_a} + u^{i\lambda}_{\alpha} \frac{\partial L_g}{\partial y^i} + J_{\mu}u^{i\lambda}_{\alpha} \frac{\partial L_g}{\partial y^i_{\mu}}\right) \end{aligned}$$

$$+ \frac{\partial L_f}{\partial h^\alpha{}_{a,\lambda}} h^\beta{}_a \partial_\beta \xi^\alpha - \xi^\alpha \left(h^\lambda{}_a \frac{\partial L_f}{\partial h^\alpha{}_a} + h^\lambda{}_{a,\mu} \frac{\partial L_f}{\partial h^\alpha{}_{a,\mu}} + u^{i\lambda}_\alpha \frac{\partial L_f}{\partial y^i} \right)$$

$$= \frac{\partial L_g}{\partial y^i_\lambda} (u^{i\beta\gamma}_\alpha \partial_{\beta\gamma} \xi^\alpha + u^{i\alpha}_\beta \partial_\alpha \xi^\beta) - \xi^\alpha J_\mu \left(u^{i\lambda}_\alpha \frac{\partial L_g}{\partial y^i_\mu} \right)$$

$$- \xi^\alpha u^{i\lambda}_\alpha \delta_i L + \frac{\partial L_f}{\partial h^\alpha{}_{a,\lambda}} h^\beta{}_a \partial_\beta \xi^\alpha - J_\mu \left(\xi^\alpha h^\lambda{}_a \frac{\partial L_f}{\partial h^\alpha{}_{a,\mu}} \right)$$

$$+ \partial_\mu \xi^\alpha h^\lambda{}_a \frac{\partial L_f}{\partial h^\alpha{}_{a,\mu}} - \xi^\alpha h^\lambda{}_a \delta_\alpha{}^a L$$

Let us introduce the symbol $\approx$ to denote equalities valid only on solutions of the field equations. Then we have

$$V^\lambda(\mathscr{L}, \xi) \approx J_\gamma \left(\frac{\partial L_g}{\partial y^i_\lambda} u^{i\beta\gamma}_\alpha \partial_\beta \xi^\alpha \right) - J_\gamma \left(\frac{\partial L_g}{\partial y^i_\lambda} u^{i\beta\gamma}_\alpha \right) \partial_\beta \xi^\alpha + \frac{\partial L_g}{\partial y^i_\lambda} u^{i\alpha}_\beta \partial_\alpha \xi^\beta$$

$$- \xi^\alpha J_\mu \left(u^{i\lambda}_\alpha \frac{\partial L_g}{\partial y^i_\mu} \right) + \partial_\beta \xi^\alpha \left(\frac{\partial L_f}{\partial h^\alpha{}_{a,\lambda}} h^\beta{}_a + \frac{\partial L_f}{\partial h^\alpha{}_{a,\beta}} h^\lambda{}_a \right)$$

$$- J_\mu \left(\xi^\alpha h^\lambda{}_a \frac{\partial L_f}{\partial h^\alpha{}_{a,\mu}} \right)$$

Due to the relation

$$\frac{\partial L_f}{\partial k^\lambda{}_{\nu\mu}} = \frac{\partial L_f}{\partial h^\lambda{}_{a,\nu}} h^\mu{}_a \tag{48}$$

and (35), the last but one term of the above expression vanishes. Moreover, turning back to the tensorial notation, one can easily verify that

$$\begin{aligned} \partial L_g / \partial y^i_\lambda u^{i\alpha\beta}_\gamma &= \pi_\gamma{}^{\alpha\beta\lambda} \\ \partial L_g / \partial y^i_\lambda u^{i\alpha}_\beta &= -\partial L_g / \partial k^\beta{}_{\alpha\lambda} - \pi_\nu{}^{\mu\alpha\lambda} k^\nu{}_{\mu\beta} \end{aligned}$$

It follows that

$$V^\lambda(\mathscr{L}, \xi) \approx J_\gamma(\pi_\alpha{}^{\beta\gamma\lambda} \partial_\beta \xi^\alpha) - J_\gamma \pi_\alpha{}^{\beta\gamma\lambda} \partial_\beta \xi^\alpha - \left(\frac{\partial L_g}{\partial k^\beta{}_{\alpha\lambda}} + \pi_\nu{}^{\mu\alpha\lambda} k^\nu{}_{\mu\beta} \right) \partial_\alpha \xi^\beta$$

$$+ \xi^\alpha J_\mu \left(\frac{\partial L_g}{\partial k^\alpha{}_{\lambda\mu}} + \pi_\nu{}^{\sigma\lambda\mu} k^\nu{}_{\sigma\alpha} \right) - J_\mu \left(\xi^\alpha h^\lambda{}_a \frac{\partial L_f}{\partial h^\alpha{}_{a,\mu}} \right)$$

$$\approx J_\gamma(\pi_\alpha{}^{\beta\gamma\lambda} \partial_\beta \xi^\alpha) + J_\mu(\xi^\alpha \pi_\nu{}^{\sigma\lambda\mu} k^\nu{}_{\sigma\alpha}) - J_\gamma \pi_\alpha{}^{\beta\gamma\lambda} \partial_\beta \xi^\alpha$$

$$
\begin{aligned}
&- \frac{\partial L_g}{\partial k^\beta{}_{\alpha\lambda}} \partial_\alpha \xi^\beta + \xi^\alpha J_\mu\left(\frac{\partial L_g}{\partial k^\alpha{}_{\lambda\mu}}\right) - J_\mu\left(\xi^\alpha h^\lambda{}_a \frac{\partial L_f}{\partial h^\alpha{}_{a,\mu}}\right) \\
&\approx J_\mu[\pi_\nu{}^{\sigma\mu\lambda}(\partial_\sigma \xi^\nu - k^\nu{}_{\sigma\alpha}\xi^\alpha)] - \delta_\alpha{}^{\beta\lambda} L\, \partial_\beta \xi^\alpha + \frac{\partial L_f}{\partial k^\alpha{}_{\beta\lambda}} \partial_\beta \xi^\alpha \\
&\quad + \xi^\alpha J_\mu\left(\frac{\partial L_g}{\partial k^\alpha{}_{\lambda\mu}}\right) - J_\mu\left(\xi^\alpha h^\lambda{}_a \frac{\partial L_f}{\partial h^\alpha{}_{a,\mu}}\right) \\
&\approx J_\mu[\pi_\nu{}^{\sigma\mu\lambda}(\partial_\sigma \xi^\nu - k^\nu{}_{\sigma\alpha}\xi^\alpha)] + \frac{\partial L_f}{\partial k^\alpha{}_{\beta\lambda}} \partial_\beta \xi^\alpha + \xi^\alpha J_\mu\left(\frac{\partial L_g}{\partial k^\alpha{}_{\lambda\mu}}\right) \\
&\quad - J_\mu\left(\xi^\alpha \frac{\partial L_f}{\partial k^\alpha{}_{\mu\lambda}}\right) \\
&\approx J_\mu[\pi_\nu{}^{\sigma\mu\lambda}(\partial_\sigma \xi^\nu - k^\nu{}_{\sigma\alpha}\xi^\alpha)] + \xi^\alpha J_\mu\left(\frac{\partial L_g}{\partial k^\alpha{}_{\lambda\mu}}\right) - \xi^\alpha J_\mu\left(\frac{\partial L_f}{\partial k^\alpha{}_{\mu\lambda}}\right) \\
&\approx J_\mu[\pi_\nu{}^{\sigma\mu\lambda}(\partial_\sigma \xi^\nu - k^\nu{}_{\sigma\alpha}\xi^\alpha)] + \xi^\alpha J_\mu(\delta_\alpha{}^{\lambda\mu} L) \\
&\approx J_\mu[\pi_\nu{}^{\sigma\mu\lambda}(\partial_\sigma \xi^\nu - k^\nu{}_{\sigma\alpha}\xi^\alpha)] \\
&\approx J_\mu[\pi_\nu{}^{\sigma\mu\lambda}(\nabla_\sigma \xi^\nu + T^\nu{}_{\alpha\sigma}\xi^\alpha)]
\end{aligned}
$$

where we have used (35), (40), and (48).

It follows that fermion fields do not contribute to the superpotential in the metric-affine theory of gravity. The stress-energy-momentum conservation law comes to the form (2), where U is the generalized Komar superpotential (4).

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